

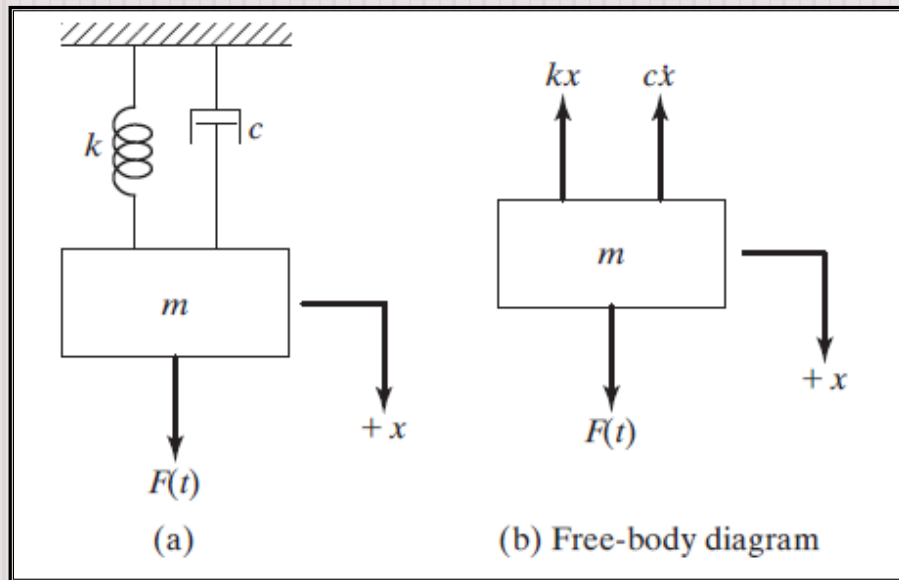
Mechanical Vibrations

Forced Vibration of a Single Degree of Freedom System

Vibration Under General Forcing Conditions

General Forcing Conditions

- Physical system



$$m \ddot{x} + c \dot{x} + kx = F(t)$$

Equation of motion

General Periodic Force

Response Under a General Periodic Force

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

$$F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} b_j \sin j\omega t$$

$$a_j = \frac{2}{\tau} \int_0^{\tau} F(t) \cos j\omega t dt, \quad j = 0, 1, 2, \dots$$

$$b_j = \frac{2}{\tau} \int_0^{\tau} F(t) \sin j\omega t dt, \quad j = 1, 2, \dots$$

General Periodic Force

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2}$$

$$m\ddot{x} + c\dot{x} + kx = a_j \cos j\omega t$$

$$m\ddot{x} + c\dot{x} + kx = b_j \sin j\omega t$$

$$x(t) = x_h(t) + x_p(t)$$

$$x_p(t) = \frac{a_0}{2k}$$

$$x_p(t) = \frac{(a_j/k)}{\sqrt{(1 - j^2 r^2)^2 + (2\zeta jr)^2}} \cos(j\omega t - \phi_j)$$

$$x_p(t) = \frac{(b_j/k)}{\sqrt{(1 - j^2 r^2)^2 + (2\zeta jr)^2}} \sin(j\omega t - \phi_j)$$

$$\phi_j = \tan^{-1} \left(\frac{2\zeta jr}{1 - j^2 r^2} \right)$$

$$r = \frac{\omega}{\omega_n}$$

General Periodic Force

$$x_p(t) = \frac{a_0}{2k} + \sum_{j=1}^{\infty} \frac{(a_j/k)}{\sqrt{(1 - j^2 r^2)^2 + (2\zeta jr)^2}} \cos(j\omega t - \phi_j) \\ + \sum_{j=1}^{\infty} \frac{(b_j/k)}{\sqrt{(1 - j^2 r^2)^2 + (2\zeta jr)^2}} \sin(j\omega t - \phi_j)$$

Non-periodic Force

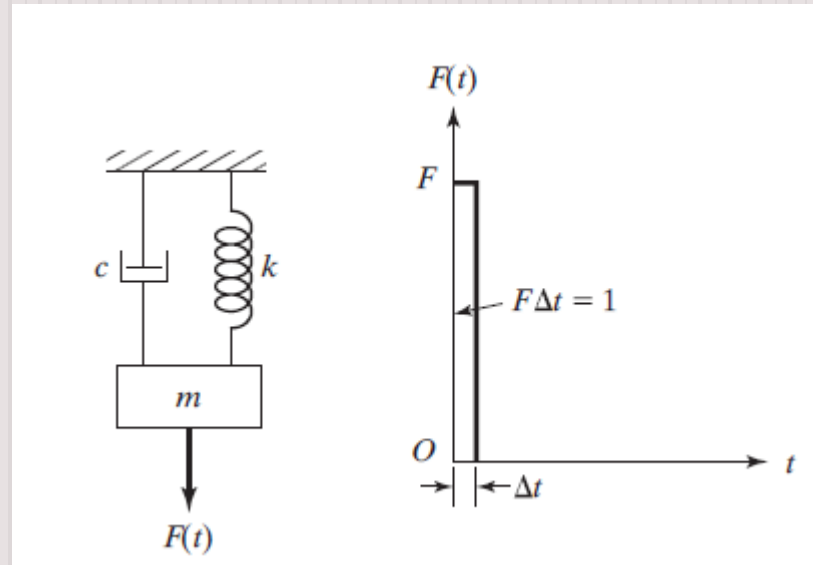
Response Under a Nonperiodic Force

Various methods can be used to find the response of the system to an arbitrary excitation. Some of these methods are as follows:

1. Representing the excitation by a Fourier integral.
2. Using the method of convolution integral.
3. Using the method of Laplace transforms.
4. Numerically integrating the equations of motion (numerical solution of differential equations).

Non-periodic Force

Response to an Impulse



$$m \ddot{x} + c \dot{x} + kx = F(t)$$

Non-periodic Force

Response to an Impulse

$$\text{Impulse} = \int_t^{t+\Delta t} F dt = m\dot{x}_2 - m\dot{x}_1$$

$$\text{Impulse} = f\Delta t = 1 = m\dot{x}(t=0) - m\dot{x}(t=0^-) = m\dot{x}_0$$

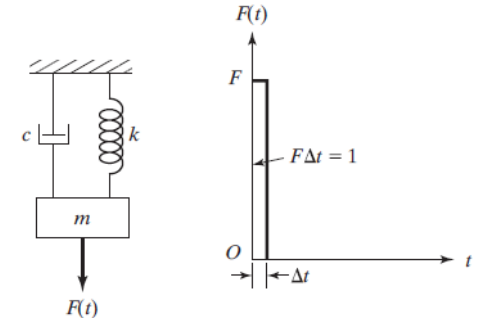
$$x(t=0) = x_0 = 0$$

$$\dot{x}(t=0) = \dot{x}_0 = \frac{1}{m}$$



$$m\ddot{x} + c\dot{x} + kx = 0$$

$$x(t) = e^{-\zeta\omega_n t} \left\{ x_0 \cos \omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} \sin \omega_d t \right\}$$



$$\zeta = \frac{c}{2m\omega_n}$$

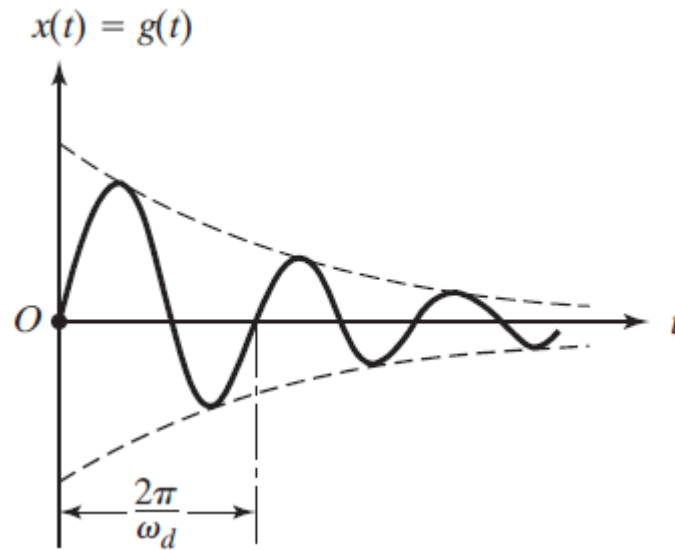
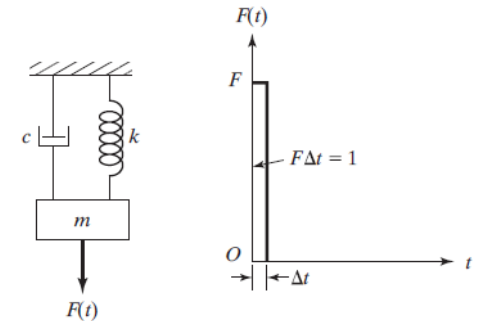
$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

Non-periodic Force

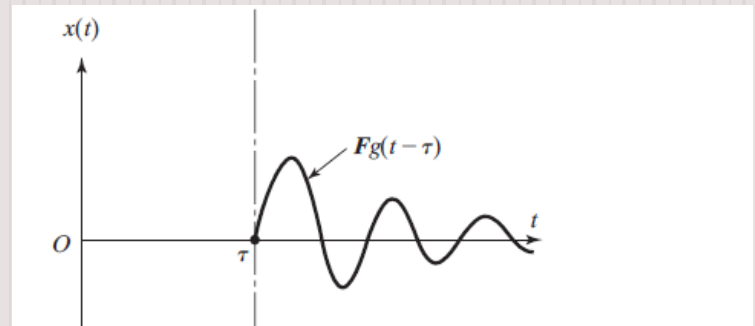
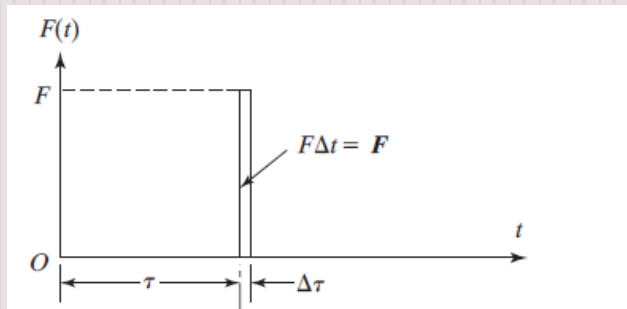
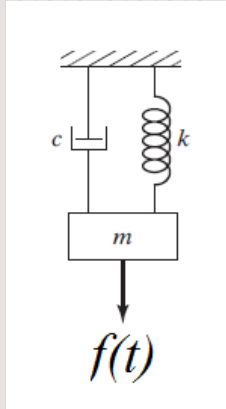
Response to an Impulse

$$x(t) = g(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d} \sin \omega_d t$$



Non-periodic Force

Response to an Impulse



$$f(t) = F \delta(t - \tau) \longrightarrow$$

$$x(t) = Fg(t - \tau)$$

Non-periodic Force

Example

In the vibration testing of a structure, an impact hammer with a load cell to measure the impact force is used to cause excitation, as shown in Fig. 4.8(a). Assuming $m = 5 \text{ kg}$, $k = 2000 \text{ N/m}$, $c = 10 \text{ N-s/m}$, and $F = 20 \text{ N-s}$, find the response of the system.

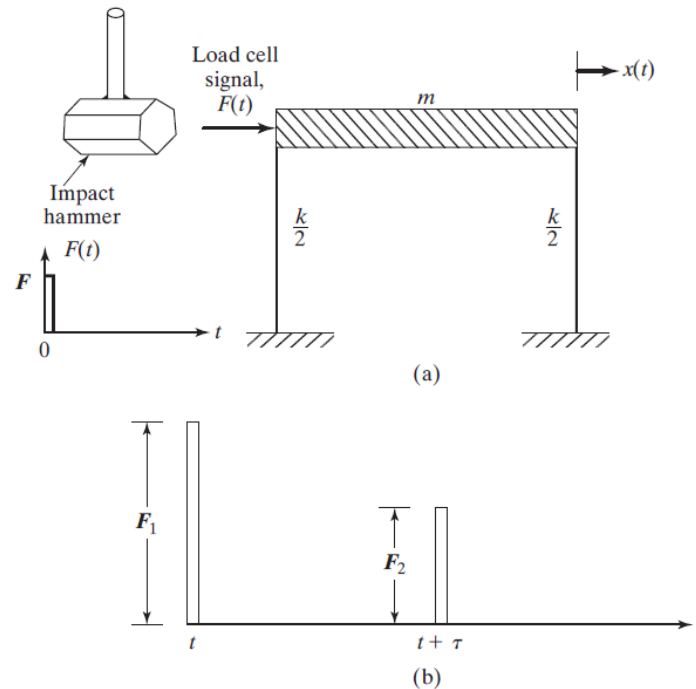


FIGURE 4.8 Structural testing using an impact hammer.

Non-periodic Force

Example

Solution: From the known data, we can compute

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{5}} = 20 \text{ rad/s}, \quad \zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{10}{2\sqrt{2000(5)}} = 0.05$$

$$\omega_d = \sqrt{1 - \zeta^2} \omega_n = 19.975 \text{ rad/s}$$

$$\begin{aligned} x_1(t) &= F \frac{e^{-\zeta \omega_n t}}{m \omega_d} \sin \omega_d t \\ &= \frac{20}{(5)(19.975)} e^{-0.05(20)t} \sin 19.975t = 0.20025 e^{-t} \sin 19.975t \text{ m} \end{aligned}$$

Non-periodic Force

Example

In many cases, providing only one impact to the structure using an impact hammer is difficult. Sometimes a second impact takes place after the first, as shown in Fig. 4.8(b), and the applied force, $F(t)$, can be expressed as

$$F(t) = F_1 \delta(t) + F_2 \delta(t - \tau)$$

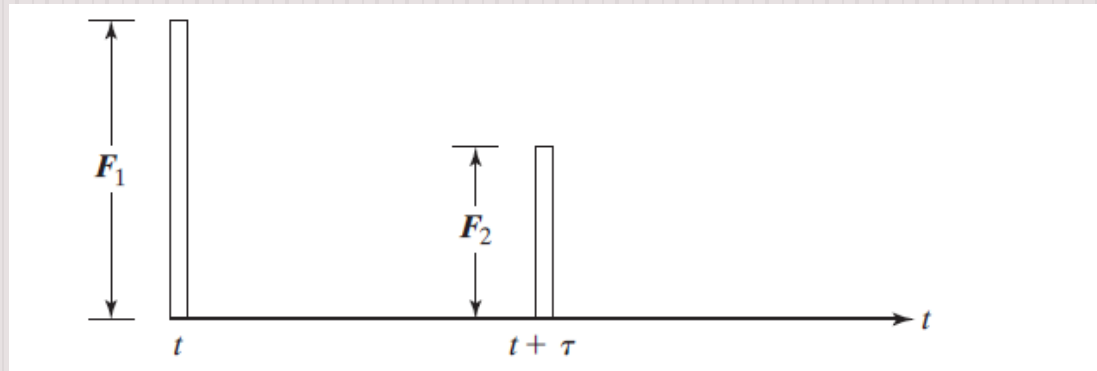
where $\delta(t)$ is the Dirac delta function and τ indicates the time between the two impacts of magnitudes F_1 and F_2 . For a structure with $m = 5$ kg, $k = 2000$ N/m, $c = 10$ N-s/m and $F(t) = 20 \delta(t) + 10 \delta(t - 0.2)$ N, find the response of the structure.

$$x_2(t) = F_2 \frac{e^{-\zeta \omega_n(t-\tau)}}{m \omega_d} \sin \omega_d(t - \tau)$$

$$\begin{aligned} x_2(t) &= \frac{10}{(5)(19.975)} e^{-0.05(20)(t-0.2)} \sin 19.975(t - 0.2) \\ &= 0.100125 e^{-(t-0.2)} \sin 19.975(t - 0.2); \quad t > 0.2 \end{aligned}$$

Non-periodic Force

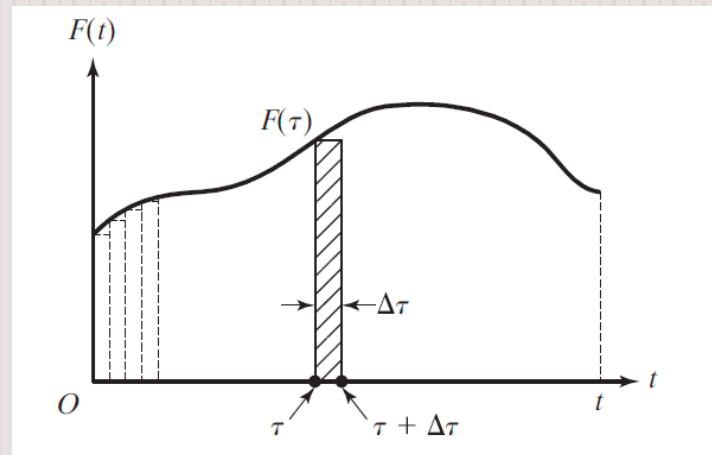
Example



$$x(t) = \begin{cases} 0.20025e^{-t} \sin 19.975t; & 0 \leq t \leq 0.2 \\ 0.20025e^{-t} \sin 19.975t + 0.100125e^{-(t-0.2)} \sin 19.975(t - 0.2); & t > 0.2 \end{cases}$$

Non-periodic Force

Response to General Forcing Conditions Based on Convolution integral method



$$\Delta x(t) = F(\tau) \Delta\tau g(t - \tau)$$



$$x(t) \simeq \sum F(\tau) g(t - \tau) \Delta\tau$$

$$x(t) = \int_0^t F(\tau) g(t - \tau) d\tau$$



$$x(t) = \frac{1}{m\omega_d} \int_0^t F(\tau) e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t - \tau) d\tau$$

Non-periodic Force

Example

A compacting machine, modeled as a single-degree-of-freedom system, is shown in Fig. 4.10(a). The force acting on the mass m (m includes the masses of the piston, the platform, and the material being compacted) due to a sudden application of the pressure can be idealized as a step force, as shown in Fig. 4.10(b). Determine the response of the system.

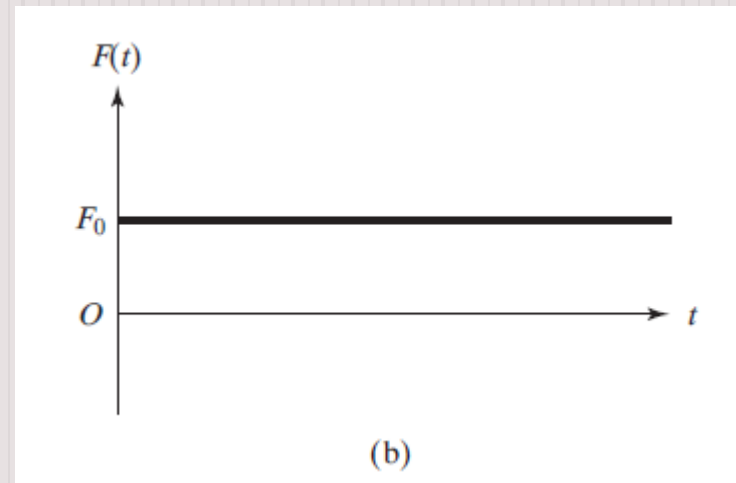
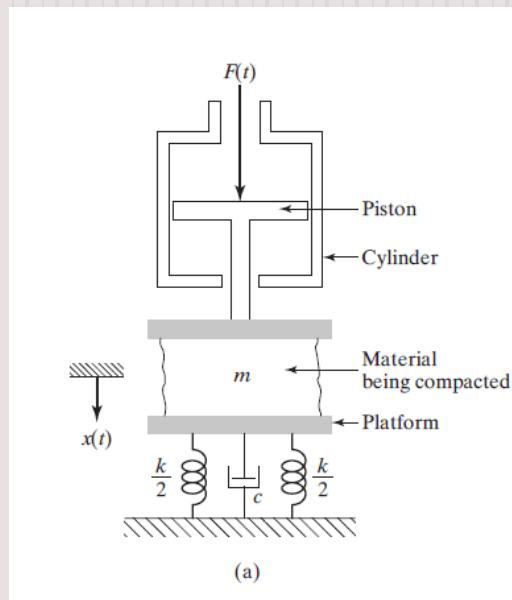
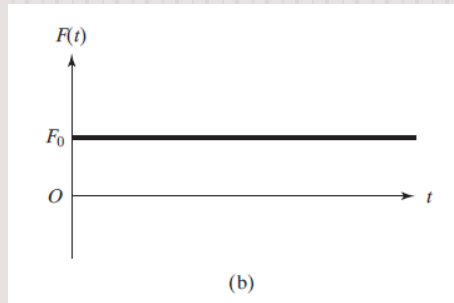
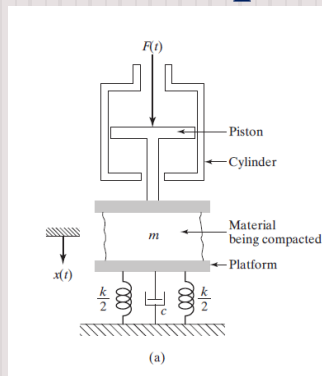


FIGURE 4.10 Step force applied to a compacting machine.

Non-periodic Force

Example



$$m \ddot{x} + c \dot{x} + kx = F(t)$$

$$x(0) = 0, \quad \dot{x}(0) = 0$$

$$F(t) = F_0$$

$$x(t) = \frac{1}{m\omega_d} \int_0^t F(\tau) e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t - \tau) d\tau$$

$$\longrightarrow x(t) = \frac{F_0}{m\omega_d} \int_0^t e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t - \tau) d\tau$$

$$= \frac{F_0}{m\omega_d} \left[e^{-\zeta\omega_n(t-\tau)} \left\{ \frac{\zeta\omega_n \sin \omega_d(t - \tau) + \omega_d \cos \omega_d(t - \tau)}{(\zeta\omega_n)^2 + (\omega_d)^2} \right\} \right]_{\tau=0}^t$$

$$= \frac{F_0}{k} \left[1 - \frac{1}{\sqrt{1 - \zeta^2}} \cdot e^{-\zeta\omega_n t} \cos(\omega_d t - \phi) \right]$$

$$\phi = \tan^{-1} \left(\frac{\zeta}{\sqrt{1 - \zeta^2}} \right)$$

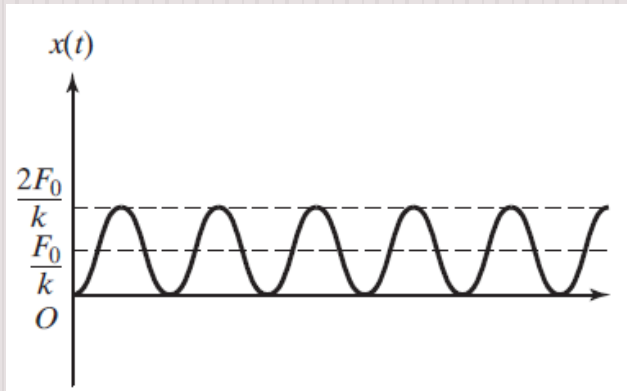
Non-periodic Force

• Example

If the system is undamped ($\zeta = 0$ and $\omega_d = \omega_n$)

$$x(t) = \frac{F_0}{k} [1 - \cos \omega_n t]$$

$$\zeta = 0$$



$$\zeta \neq 0$$

